

DEGREE SEQUENCES FOR GRAPHS WITH LOOPS

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ABSTRACT. This paper considers graphs, without multiple edges, in which there is at most one loop at each vertex. We give Erdős–Gallai type theorems for such graphs and we show how they relate to bipartite graphs in which the two parts have the same degree sequence.

1. GRAPHS WITH LOOPS

Let us first specify the object of our investigation.

Definition 1. By a *graph-with-loops* we mean a graph, without multiple edges, in which there is at most one loop at each vertex.

Our main interest in this paper is to make the connection between graph-with-loops and bipartite graphs. Recall that for every simple graph G , there is a natural associated bipartite simple graph \hat{G} called the *bipartite double-cover* of G . For convenience in the following discussion, assume that G is connected. From a practical point of view, \hat{G} can be constructed as follows: remove from G the minimum possible number of edges so as to make the resulting graph bipartite. Take two copies G_1, G_2 of this bipartite graph, and two copies of the removed edges, and reattach the edges to their original vertices but in such a way that each attached edge joins G_1 with G_2 . The resulting bipartite graph \hat{G} is independent of the choices made in its construction. From a theoretical point of view, \hat{G} can be conceived in (at least) two natural ways:

- (1) \hat{G} is the *tensor product* $G \times K_2$ of G with the connected graph with 2 vertices; the vertex set of $G \times K_2$ is the Cartesian product of the vertices of G and K_2 , there are edges in $G \times K_2$ between $(a, 0)$ and $(b, 1)$ and between $(a, 1)$ and $(b, 0)$ if and only if there is an edge in G between a and b .
- (2) \hat{G} has a topological definition as a covering space of G . Give G the obvious topology, and let H denote the subgroup of the fundamental group $\pi_1(G)$ composed of loops of even length, where *length* is defined by taking each edge to have length one. Clearly H has index two in $\pi_1(G)$, so it is a normal subgroup. Thus, by standard covering space theory, there is a two-fold normal covering space \hat{G} of G , and by construction, \hat{G} is bipartite; see [7, Chapter 1.3].

Now consider the situation where G is a graph-with-loops. Here the two constructions described above do not produce the same graph. The distinction is that in the tensor product, each loop in G produces just one edge in $G \times K_2$, while in the covering space, each loop lifts to two edges. Figure 1 shows the constructions for the complete graph-with-loops G on three vertices. In the tensor product construction, $G \times K_2$ is not a covering of G in the topological sense, and in particular, the vertices in $G \times K_2$ do not have the same degree as

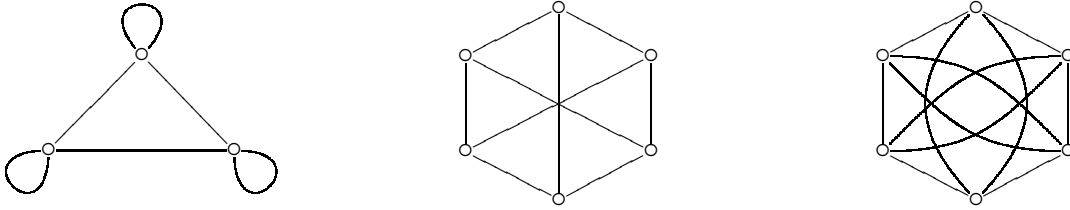


FIGURE 1. The complete graph-with-loops on three vertices, and its two “bi-partite covers”

those in G . In the topological construction, the covering space is not a simple graph, but a multigraph, however its vertex degrees do have the same degree as those in G . We will argue below that the tensor product construction is the appropriate concept for graphs-with-loops.

In the following we will need to refer to the Erdős–Gallai Theorem, which we recall for convenience.

Erdős–Gallai Theorem. *A sequence $\underline{d} = (d_1, \dots, d_n)$ of nonnegative integers in decreasing order is graphic if and only if its sum is even and, for each integer k with $1 \leq k \leq n$,*

$$(EG) \quad \sum_{i=1}^k d_i \leq k(k-1) + \sum_{i=k+1}^n \min\{k, d_i\}.$$

Note that graphs-with-loops are a special family of multigraphs [5] and that for multigraphs, the *degree* of a vertex is usually taken to be the number of edges incident to the vertex, with loops counted twice. We have the following straightforward generalisation of the Erdős–Gallai Theorem. We postpone the proof of this theorem to the final section.

Theorem 1. *Let $\underline{d} = (d_1, \dots, d_n)$ be a sequence of nonnegative integers in decreasing order. Then \underline{d} is the sequence of vertex degrees of a graph-with-loops if and only if its sum is even and, for each integer k with $1 \leq k \leq n$,*

$$(1) \quad \sum_{i=1}^k d_i \leq k(k+1) + \sum_{i=k+1}^n \min\{k, d_i\}.$$

Despite the above result, we claim that in the context of graphs-with-loops, a different definition of degree is more appropriate. We introduce the following definition.

Definition 2. For a graph-with-loops, the *reduced degree* of a vertex is taken to be the number of edges incident to the vertex, with loops counted once.

So, for example, in the complete graph-with-loops G on three vertices, shown on the left in Figure 1, the vertices each have reduced degree three. Notice the vertices in the tensor product $G \times K_2$ also have the same degrees as the reduced degrees of the vertices of G . In our view, the tensor product construction is the appropriate concept of bipartite double cover for graphs-with-loops; as we discussed above, it does not produce a covering space, in the topological sense, but it does produce a bipartite graph, and the vertex degrees are

preserved provided we use the notion of reduced degree. We have the following Erdős–Gallai type result; the proof is given in the final section.

Theorem 2. *Let $\underline{d} = (d_1, \dots, d_n)$ be a sequence of nonnegative integers in decreasing order. Then \underline{d} is the sequence of reduced degrees of the vertices of a graph-with-loops if and only if for each integer k with $1 \leq k \leq n$,*

$$(2) \quad \sum_{i=1}^k d_i \leq k^2 + \sum_{i=k+1}^n \min\{k, d_i\}.$$

Let us say that a finite sequence \underline{d} of nonnegative integers is *bipartite graphic* if the pair $(\underline{d}, \underline{d})$ can be realized as the degree sequences of the parts of a bipartite simple graph; such sequences have been considered in [1]. The utility of the notion of reduced degree is apparent in the following result.

Corollary 1. *A sequence $\underline{d} = (d_1, \dots, d_n)$ of nonnegative integers in decreasing order is the sequence of reduced degrees of the vertices of a graph-with-loops if and only if \underline{d} is bipartite graphic.*

Proof. If \underline{d} is the sequence of reduced degrees of the vertices of a graph-with-loops G , then forming the tensor product $G \times K_2$ we obtain a bipartite graph having degree sequence $(\underline{d}, \underline{d})$. Conversely, if \underline{d} is bipartite graphic, then by the Gale–Ryser Theorem [6, 10], for each k with $1 \leq k \leq n$,

$$\sum_{i=1}^k d_i \leq \sum_{i=1}^n \min\{k, d_i\} \leq k^2 + \sum_{i=k+1}^n \min\{k, d_i\},$$

and so by Theorem 2, \underline{d} is the sequence of reduced degrees of the vertices of a graph-with-loops. \square

2. SOME REMARKS

Remark 1. Corollary 1 has two consequences. First, from Theorem 2 and Corollary 1, condition (2) gives an Erdős–Gallai type condition for a sequence to be bipartite graphic, which is analogous to the Gale–Ryser condition. Secondly, consider a bipartite graphic sequence \underline{d} . So $(\underline{d}, \underline{d})$ can be realised as the degree sequences of the parts of a bipartite graph \hat{G} . Corollary 1 shows that this can be done in a symmetric manner, in that \hat{G} is a tensor product $\hat{G} = G \times K_2$, and in particular there is an involutive graph automorphism that interchanges the two parts of \hat{G} .

Remark 2. It is clear from the discussion in Section 1 that if a sequence (d_1, \dots, d_n) is graphic, then the sequence $(d_1 + 1, d_2 + 1, \dots, d_n + 1)$ is bipartite graphic. Note that the converse is not true; for example, $(4, 4, 2, 2)$ is bipartite graphic, while $(3, 3, 1, 1)$ is not graphic.

Remark 3. There are several results in the literature of the following kind: if \underline{d} is graphic, and if \underline{d}' is obtained from \underline{d} using a particular construction, then \underline{d}' is also graphic. The Kleitman–Wang Theorem is of this kind [8]. Another useful result is implicit in Choudum’s proof [4] of the Erdős–Gallai Theorem: If a decreasing sequence $\underline{d} = (d_1, \dots, d_n)$ of positive

integers is graphic, then so is the sequence \underline{d}' obtained by reducing both d_1 and d_n by one. Analogously, our proofs of Theorems 1 and 2, which are modelled on Choudum's proof, also establish the following result: If a decreasing sequence $\underline{d} = (d_1, \dots, d_n)$ of positive integers is bipartite graphic, then so is the sequence \underline{d}' obtained by reducing both d_1 and d_n by one.

Remark 4. There are other facts for graphs that generalise easily to graphs-with-loops. For example, if $\underline{d} = (d_1, \dots, d_n)$ has a realization as a graph G , then one can consider G as a subgraph of the complete graph K_n . Considering the complement of G in K_n , one obtains the well known result: If (d_1, \dots, d_n) is graphic, then so too is $(n - 1 - d_n, n - 1 - d_{n-1}, \dots, n - 1 - d_1)$. Analogously, by replacing the complete graph on n vertices by the complete graph-with-loops on n vertices, one immediately obtains the following equivalent result: If a sequence $\underline{d} = (d_1, \dots, d_n)$ of nonnegative integers is bipartite graphic, then so is the sequence $\underline{d}' = (n - d_n, n - d_{n-1}, \dots, n - d_1)$.

Remark 5. For criteria for sequences to be realized by multigraphs, see [9]. There are many other recent papers on graphic sequences, see for example [12, 11, 13, 14, 2, 3].

3. PROOFS OF THEOREMS 1 AND 2

The following two proofs are modelled on Choudum's proof of the Erdős–Gallai Theorem [4].

Proof of Theorem 1. For the proof of necessity, first note that for every graph-with-loops with vertex degree sequence \underline{d} , the sum $\sum_{i=1}^n d_i$ is twice the number of edges, so it is even. Now consider the set S comprised of the first k vertices. The left hand side of (1) is the number of half-edges incident to S , with each loop counting as two. On the right hand side, $k(k+1)$ is the number of half-edges in the complete graph-with-loops on S , again with each loop counting as two, while $\sum_{i=k+1}^n \min\{k, d_i\}$ is the maximum number of edges that could join vertices in S to vertices outside S . So (1) is obvious.

The proof of sufficiency is by induction on $\sum_{i=1}^n d_i$. It is obvious for $\sum_{i=1}^n d_i = 2$. Suppose that we have a decreasing sequence $\underline{d} = (d_1, \dots, d_n)$ of positive integers which has even sum and satisfies (1). As in Choudum's proof of the Erdős–Gallai Theorem, consider the sequence \underline{d}' obtained by reducing both d_1 and d_n by 1. Let \underline{d}'' denote the sequence obtained by reordering \underline{d}' so as to be decreasing.

Suppose that \underline{d}'' satisfies (1) and hence by the inductive hypothesis, there is a graph-with-loops G' that realizes \underline{d}'' . We will show how \underline{d} can be realized. Let the vertices of G' be labelled v_1, \dots, v_n . [Note that v_n may be an isolated vertex]. If there is no edge in G' connecting v_1 to v_n , then add one; this gives a graph-with-loops G that realizes \underline{d} . So it remains to treat the case where there is an edge in G' connecting v_1 to v_n . If there is no loop at either v_1 or v_n , remove the edge between v_1 and v_n , and add loops at both v_1 and v_n .

Now, for the moment, let us assume there is a loop in G' at v_1 . Applying the hypothesis to \underline{d} , using $k = 1$ gives

$$d_1 \leq 2 + \sum_{i=2}^n \min\{k, d_i\} \leq n + 1,$$

and so $d_1 - 3 < n - 1$. Now in G' , the degree of v_1 is $d_1 - 1$ and so apart from the loop at v_1 , there are a further $d_1 - 3$ edges incident to v_1 . So in G' , there is some vertex $v_i \neq v_1$,

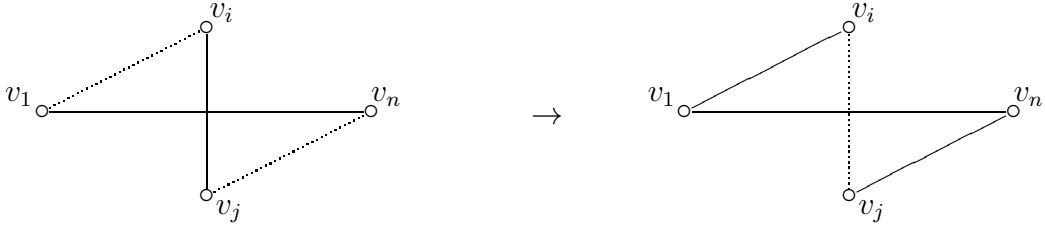


FIGURE 2.

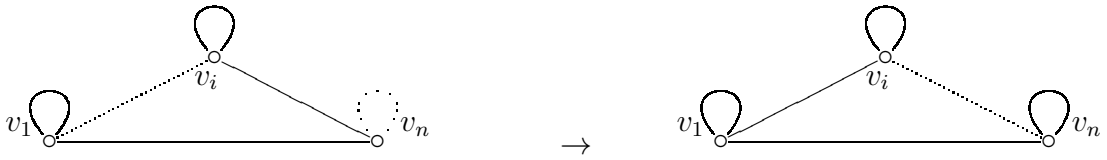


FIGURE 3.

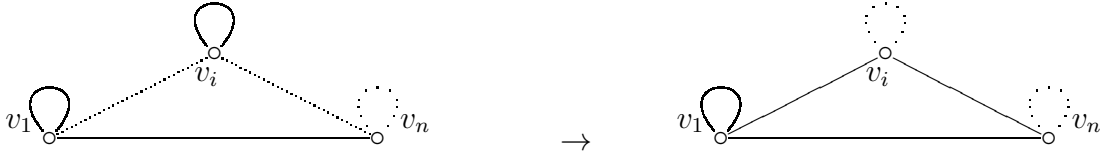


FIGURE 4.

for which there is no edge from v_1 to v_i . Note that $d'_i > d'_n$. If there is a loop in G' at v_n , or if there is no loop at v_i nor at v_n , then there is a vertex v_j such that there is an edge in G' from v_i to v_j , but there is no edge from v_j to v_n . Now remove the edge $v_i v_j$, and put in edges from v_1 to v_i , and from v_j to v_n , as in Figure 2. This gives a graph-with-loops G that realizes \underline{d} . If there is no loop in G' at v_n , but there is a loop at v_i , we consider the two cases according to whether or not there is an edge between v_i and v_n . If there is an edge between v_i and v_n , then remove this edge, add an edge $v_1 v_i$ and add a loop at v_n , as in Figure 3. If there is no edge between v_i and v_n , add edges $v_1 v_i$ and $v_i v_n$ and remove the loop at v_i , as in Figure 4. In either case, we again obtain a graph-with-loops G that realizes \underline{d} .

Finally, assume there is no loop in G' at v_1 , but there is a loop in G' at v_n . So, apart from the loop, there are a further $d_n - 3$ edges incident to v_n . Since $d_1 \geq d_n$, we have $d_1 - 1 > d_n - 3$, and so there is a vertex v_i such that there is an edge in G' from v_1 to v_i , but there is no edge from v_i to v_n . Remove the edge $v_1 v_i$, put in an edge $v_i v_n$ and add a loop at v_1 , as in Figure 5. The resulting graph-with-loops G realizes \underline{d} .

It remains to show that \underline{d}'' satisfies (1). Define m as follows: if the d_i are all equal, put $m = n - 1$, otherwise, define m by the condition that $d_1 = \dots = d_m$ and $d_m > d_{m+1}$. We have $d''_i = d_i$ for all $i \neq m, n$, while $d''_m = d_m - 1$ and $d''_n = d_n - 1$. Consider condition (1) for



FIGURE 5.

\underline{d}'' :

$$(3) \quad \sum_{i=1}^k d_i'' \leq k(k+1) + \sum_{i=k+1}^n \min\{k, d_i''\}.$$

For $m \leq k < n$, we have $\sum_{i=1}^k d_i'' = \sum_{i=1}^k d_i - 1$, while $\sum_{i=k+1}^n \min\{k, d_i''\} \geq \sum_{i=k+1}^n \min\{k, d_i\} - 1$, and so (3) holds. For $k = n$, $\sum_{i=1}^k d_i'' = \sum_{i=1}^k d_i - 2 < k(k+1)$, and so (3) again holds. For $k < m$, first note that if $d_k \leq k+1$, then $\sum_{i=1}^k d_i'' = \sum_{i=1}^k d_i \leq k(k+1) \leq k(k+1) + \sum_{i=k+1}^n \min\{k, d_i''\}$. So it remains to deal with the case where $k < m$ and $d_k > k+1$. We have

$$\sum_{i=1}^k d_i'' = \sum_{i=1}^k d_i \leq k(k+1) + \sum_{i=k+1}^n \min\{k, d_i\}.$$

Notice that as $d_i = d_i''$ except for $i = m, n$, we have $\min\{k, d_i''\} = \min\{k, d_i\}$ except possibly for $i = m, n$. In fact, as $k < m$, we have $d_m = d_k > k+1$ and so $\min\{k, d_m\} = k = \min\{k, d_m''\}$. Hence $\sum_{i=k+1}^n \min\{k, d_i''\} \geq \sum_{i=k+1}^n \min\{k, d_i\} - 1$. Thus, in order to establish (3), it suffices to show that $\sum_{i=1}^k d_i < k(k+1) + \sum_{i=k+1}^n \min\{k, d_i\}$. Suppose instead that $\sum_{i=1}^k d_i = k(k+1) + \sum_{i=k+1}^n \min\{k, d_i\}$. We have

$$kd_m = \sum_{i=1}^k d_i = k(k+1) + \sum_{i=k+1}^n \min\{k, d_i\}$$

and so

$$d_m = (k+1) + \frac{1}{k} \sum_{i=k+1}^n \min\{k, d_i\}.$$

Then

$$\sum_{i=1}^{k+1} d_i = (k+1)d_m = (k+1)^2 + \frac{k+1}{k} \sum_{i=k+1}^n \min\{k, d_i\}.$$

We have $d_{k+1} = d_m > k+1$ and so $\min\{k, d_{k+1}\} = k$. Note that $\sum_{i=k+2}^n \min\{k, d_i\} \neq 0$ as $k+2 \leq n$, since $k < m \leq n-1$. So

$$\sum_{i=1}^{k+1} d_i = (k+1)^2 + (k+1) + \frac{k+1}{k} \sum_{i=k+2}^n \min\{k, d_i\} > (k+1)(k+2) + \sum_{i=k+2}^n \min\{k, d_i\},$$

contradicting (1). Hence \underline{d}'' satisfies (3), as claimed. This completes the proof. \square

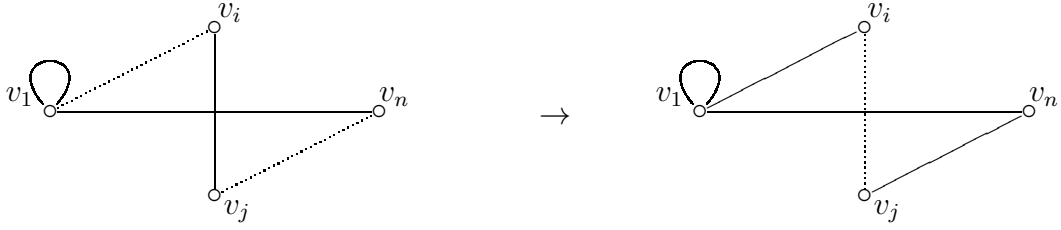


FIGURE 6.

Proof of Theorem 2. The proof mimics the above proof of Theorem 1. For the proof of necessity, consider the set S comprised of the first k vertices. The left hand side of (1) is the number of half-edges incident to S , with each loop counting as one. On the right hand side, k^2 is the number of half-edges in the complete graph-with-loops on S , again with each loop counting as one, while $\sum_{i=k+1}^n \min\{k, d_i\}$ is the maximum number of edges that could join vertices in S to vertices outside S . So (1) is obvious.

Conversely, suppose that $\underline{d} = (d_1, \dots, d_n)$ verifies (2) and consider the sequence \underline{d}' obtained by reducing both d_1 and d_n by 1. Let \underline{d}'' denote the sequence obtained by reordering \underline{d}' so as to be decreasing. Suppose that \underline{d}'' satisfies (2) and hence by the inductive hypothesis, there is a graph-with-loops G' that realizes \underline{d}'' . We will show how \underline{d} can be realized. Let the vertices of G' be labelled v_1, \dots, v_n . If there is no edge in G' connecting v_1 to v_n , then add one; this gives a graph-with-loops G that realizes \underline{d} . Similarly, if there is no loop at either v_1 or v_n , just add loops at both v_1 and v_n . So it remains to treat the case where there is an edge in G' connecting v_1 to v_n , and at least one of the vertices v_1, v_n has a loop.

Now, for the moment, let us assume there is a loop in G' at v_1 . Applying the hypothesis to \underline{d} , using $k = 1$ gives

$$d_1 \leq 1 + \sum_{i=2}^n \min\{k, d_i\} \leq n,$$

and so $d_1 - 2 < n - 1$. Now in G' , the degree of v_1 is $d_1 - 1$ and so apart from the loop at v_1 , there are a further $d_1 - 2$ edges incident to v_1 . So in G' , there is some vertex $v_i \neq v_1$, for which there is no edge from v_1 to v_i . Note that $d'_i > d'_n$. If there is a loop in G' at v_n , or if there is no loop at v_i nor at v_n , then there is a vertex v_j such that there is an edge in G' from v_i to v_j , but there is no edge from v_j to v_n . Now remove the edge $v_i v_j$, and put in edges from v_1 to v_i , and from v_j to v_n , as in Figure 6. This gives a graph-with-loops G that realizes \underline{d} . If there is no loop in G' at v_n , but there is a loop at v_i , remove the loop at v_i , add the edge $v_1 v_i$ and add a loop at v_n , as in Figure 7.

Finally, assume there is no loop in G' at v_1 , but there is a loop in G' at v_n . So, apart from the loop, there are a further $d_n - 2$ edges incident to v_n . Since $d_1 \geq d_n$, we have $d_1 - 1 > d_n - 2$, and so there is a vertex v_i such that there is an edge in G' from v_1 to v_i , but there is no edge from v_i to v_n . Note that $d'_i > d'_n$, so as there is a loop in G' at v_n , there is a vertex v_j such that there is an edge in G' from v_i to v_j , but there is no edge from v_j to v_n . Now remove the loop at v_n and the edge $v_i v_j$, and put edges $v_j v_n$ and $v_i v_n$ and add a loop at v_1 , as in Figure 8. This gives a graph-with-loops G that realizes \underline{d} .



FIGURE 7.

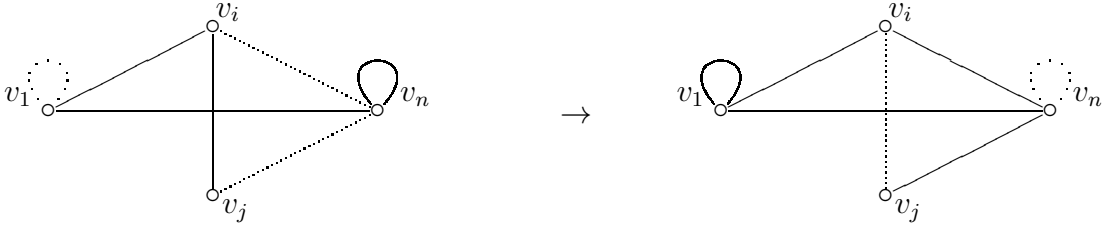


FIGURE 8.

It remains to show that \underline{d}'' satisfies (2). Define m as follows: if the d_i are all equal, put $m = n - 1$, otherwise, define m by the condition that $d_1 = \dots = d_m$ and $d_m > d_{m+1}$. We have $d_i'' = d_i$ for all $i \neq m, n$, while $d_m'' = d_m - 1$ and $d_n'' = d_n - 1$. Consider condition (2) for \underline{d}'' :

$$(4) \quad \sum_{i=1}^k d_i'' \leq k^2 + \sum_{i=k+1}^n \min\{k, d_i''\}.$$

For $m \leq k < n$, we have $\sum_{i=1}^k d_i'' = \sum_{i=1}^k d_i - 1$, while $\sum_{i=k+1}^n \min\{k, d_i''\} \geq \sum_{i=k+1}^n \min\{k, d_i\} - 1$, and so (4) holds. For $k = n$, $\sum_{i=1}^k d_i'' = \sum_{i=1}^k d_i - 2 < k^2$, and so (4) again holds. For $k < m$, first note that if $d_k \leq k$, then $\sum_{i=1}^k d_i'' = \sum_{i=1}^k d_i \leq k^2 \leq k^2 + \sum_{i=k+1}^n \min\{k, d_i''\}$. So it remains to deal with the case where $k < m$ and $d_k > k$. We have

$$\sum_{i=1}^k d_i'' = \sum_{i=1}^k d_i \leq k^2 + \sum_{i=k+1}^n \min\{k, d_i\}.$$

Notice that as $d_i = d_i''$ except for $i = m, n$, we have $\min\{k, d_i''\} = \min\{k, d_i\}$ except possibly for $i = m, n$. In fact, as $k < m$, we have $d_m = d_k > k$ and $d_m'' = d_m - 1 \geq k$ and so $\min\{k, d_m\} = k = \min\{k, d_m''\}$. Hence $\sum_{i=k+1}^n \min\{k, d_i''\} \geq \sum_{i=k+1}^n \min\{k, d_i\} - 1$. Thus, in order to establish (4), it suffices to show that $\sum_{i=1}^k d_i < k^2 + \sum_{i=k+1}^n \min\{k, d_i\}$. Suppose instead that $\sum_{i=1}^k d_i = k^2 + \sum_{i=k+1}^n \min\{k, d_i\}$. We have

$$kd_m = \sum_{i=1}^k d_i = k^2 + \sum_{i=k+1}^n \min\{k, d_i\}$$

and so

$$d_m = k + \frac{1}{k} \sum_{i=k+1}^n \min\{k, d_i\}.$$

Then

$$\sum_{i=1}^{k+1} d_i = (k+1)d_m = k(k+1) + \frac{k+1}{k} \sum_{i=k+1}^n \min\{k, d_i\}.$$

We have $d_{k+1} = d_m > k$ and so $\min\{k, d_{k+1}\} = k$. Note that $\sum_{i=k+2}^n \min\{k, d_i\} \neq 0$ as $k+2 \leq n$, since $k < m \leq n-1$. So

$$\sum_{i=1}^{k+1} d_i = k(k+1) + (k+1) + \frac{k+1}{k} \sum_{i=k+2}^n \min\{k, d_i\} > (k+1)^2 + \sum_{i=k+2}^n \min\{k, d_i\},$$

contradicting (2). Hence \underline{d}'' satisfies (4), as claimed. This completes the proof. \square

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REFERENCES

1. Noga Alon, Sonny Ben-Shimon, and Michael Krivelevich, *A note on regular Ramsey graphs*, J. Graph Theory **64** (2010), no. 3, 244–249.
2. Michael D. Barrus, Stephen G. Hartke, Kyle F. Jao, and Douglas B. West, *Length thresholds for graphic lists given fixed largest and smallest entries and bounded gaps*, Discrete Math. **312** (2012), no. 9, 1494–1501.
3. Grant Cairns and Stacey Mendan, *An improvement of a result of Zverovich–Zverovich*, preprint.
4. S. A. Choudum, *A simple proof of the Erdős–Gallai theorem on graph sequences*, Bull. Austral. Math. Soc. **33** (1986), no. 1, 67–70.
5. Reinhard Diestel, *Graph theory*, fourth ed., Graduate Texts in Mathematics, vol. 173, Springer, Heidelberg, 2010.
6. David Gale, *A theorem on flows in networks*, Pacific J. Math. **7** (1957), 1073–1082.
7. Allen Hatcher, *Algebraic topology*, Cambridge University Press, Cambridge, 2002.
8. D. J. Kleitman and D. L. Wang, *Algorithms for constructing graphs and digraphs with given valences and factors*, Discrete Math. **6** (1973), 79–88.
9. Dirk Meierling and Lutz Volkmann, *A remark on degree sequences of multigraphs*, Math. Methods Oper. Res. **69** (2009), no. 2, 369–374.
10. H. J. Ryser, *Combinatorial properties of matrices of zeros and ones*, Canad. J. Math. **9** (1957), 371–377.
11. Amitabha Tripathi and Himanshu Tyagi, *A simple criterion on degree sequences of graphs*, Discrete Appl. Math. **156** (2008), no. 18, 3513–3517.
12. Amitabha Tripathi and Sujith Vijay, *A note on a theorem of Erdős & Gallai*, Discrete Math. **265** (2003), no. 1-3, 417–420.
13. Jian-Hua Yin, *Conditions for r -graphic sequences to be potentially $K_{m+1}^{(r)}$ -graphic*, Discrete Math. **309** (2009), no. 21, 6271–6276.
14. Jian-Hua Yin and Jiong-Sheng Li, *Two sufficient conditions for a graphic sequence to have a realization with prescribed clique size*, Discrete Math. **301** (2005), no. 2-3, 218–227.

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